

# SYSTOLIC GROWTH OF LINEAR GROUPS

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**ABSTRACT.** We prove that the residual girth of any finitely generated linear group is at most exponential. This means that the smallest finite quotient in which the  $n$ -ball injects has at most exponential size. If the group is also not virtually nilpotent, it follows that the residual girth is precisely exponential.

## 1. INTRODUCTION

Let  $\Gamma$  be a group with a finite generating subset  $S$ , and  $|\cdot|_S$  the corresponding word length. We assume for convenience that  $S$  is symmetric and contains the unit, so that  $S^n$  is equal to the  $n$ -ball. The following three functions are attached to  $(\Gamma, S)$ :

- the growth: the cardinal  $b_{\Gamma,S}(n)$  of  $S^n$ ;
- the systolic growth: the function  $\sigma_{\Gamma,S}$  mapping  $n$  to the smallest  $k$  such that some subgroup  $H$  of index  $k$  contains no nontrivial element of the  $n$ -ball; if no such  $k$  exists, we define it as  $+\infty$ ;
- the residual girth, or normal systolic growth  $\sigma'_{\Gamma,S}$ : same definition, with the additional requirement that  $H$  is normal.

The growth is always defined and is at most exponential, while the systolic growth and residual girth take finite values if and only if  $\Gamma$  is residually finite, and in this case they can be larger than exponential, as the example in [BSe] show. Furthermore, we have the obvious inequalities

$$b_{\Gamma,S}(n) \leq \sigma_{\Gamma,S}(2n+1) \leq \sigma'_{\Gamma,S}(2n+1).$$

The asymptotic behavior of these functions, for finitely generated groups, does not depend on the finite generating subset.

A simple example for the residual girth grows strictly faster than the systolic growth is the case of the integral Heisenberg group, for which the growth and systolic growth behaves as  $n^4$  while the residual girth grows as  $n^6$  (see [BSt, C]). Also the systolic growth may grow faster than the growth and actually can grow arbitrarily fast. We show here that in linear groups, this is not the case.

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*Date:* August 21, 2014.

*2010 Mathematics Subject Classification.* Primary 20E26; Secondary 11C99, 13B25, 20F65.

K.B. is supported in part by NSF DMS-1405609; Y.C. is supported in part by ANR GSG 12-BS01-0003-01.

**Theorem 1.1.** *Assume that  $\Gamma$  admits a faithful finite-dimensional representation over a field (or a product of fields). Then the residual girth (and hence the systolic growth) of  $\Gamma$  are at most exponential. In particular, if  $\Gamma$  is not virtually nilpotent, then its residual girth and its systolic growth are exponential.*

Such a result was asserted by Gromov [G, p.334] for subgroups of  $\mathrm{SL}_d(\mathbf{Z})$ , under some technical superfluous additional assumption (non-existence of nontrivial unipotent elements).

The proof of Theorem 1.1 consists in finding small enough quotient fields of the ring of entries, while ensuring that the  $n$ -ball is mapped injectively. The argument can be simplified in case  $\Gamma \subset \mathrm{GL}_d(\mathbf{Q})$ , since then reduction modulo  $p$  for all  $p$  large enough work with no further effort; in this case the finite quotients are explicit, while in the general case we only find a suitable quotient field using a counting argument.

*Example 1.2.* The group  $\mathbf{Z} \wr \mathbf{Z}$  has an exponential residual girth. Another example is  $(\mathbf{Z}/6\mathbf{Z}) \wr \mathbf{Z}$ , which is linear over a product of 2 fields, but not over a single field.

*Remark 1.3.* Closely related functions are the residual finiteness growth, which maps  $n$  to the smallest number  $s_{\Gamma,S}(n)$  such that for every  $g \in S^n \setminus \{1\}$ , there is a finite index subgroup of  $\Gamma$  avoiding  $g$ , and  $s_{\Gamma,S}^{\triangleleft}(n)$  defined in the same way with only normal finite index subgroups. For finitely generated group that are linear over a field, a polynomial upper bound for these functions is established in [BM], and in the case of higher rank arithmetic groups, the precise behavior is obtained in [BK]: for instance, for  $\mathrm{SL}_d(\mathbf{Z})$  for  $d \geq 3$ , the normal residual finiteness growth grows as  $n^{d^2-1}$ .

## 2. PRELIMINARIES ON POLYNOMIALS OVER FINITE FIELDS

**Lemma 2.1.** *Let  $F$  be a finite field with  $q$  elements. Given an integer  $n \geq 1$ , the number of irreducible monic polynomials of degree  $n$  in  $F[t]$  is  $\leq q^n/n$  and  $\geq (q^n - q^{n-1})/n$ .*

*Proof.* The case  $n = 1$  being trivial, we can assume  $n \geq 1$ . By Gauss' formula this number  $N_q(n)$  is equal to  $(1/n) \sum_{d|n} \mu(n/d) q^d$ , where  $\mu$  is Möbius' function. Let  $p > 1$  be the smallest prime divisor of  $n$ . Then

$$\begin{aligned} \sum_{d|n} \mu(n/d) q^d &= q^n - q^{n/p} + \sum_{d|n, d > p} \mu(n/d) q^d \leq q^n - q^{n/p} + \sum_{d|n, d > p} q^d \\ &\leq q^n - q^{n/p} + \sum_{k=0}^{n/p-1} q^k \leq q^n \end{aligned}$$

A similar argument shows that  $nN_q(n) \geq q^n - q^{1+n/p}$ , which is  $\geq q^n - q^{n-1}$  if  $n \geq 3$ ; the cases  $n \leq 2$  being trivial.  $\square$

**Lemma 2.2.** *Let  $F$  be a field with  $q$  elements. Let  $P \in F[t]$  be a nonzero polynomial of degree  $\leq n$ . Then  $P$  survives in a quotient field of  $F[t]$  of cardinal  $\leq 2nq$ .*

*Proof.* Let  $m \geq 1$  be the largest number such that every irreducible polynomial of degree  $m - 1$  divides  $P$ . Let us check that  $q^m \leq 2nq$ ; the case  $m = 1$  being trivial, we assume  $m \geq 2$ . By Lemma 2.1, there are  $\geq (q^{m-1} - q^{m-2})/(m - 1)$  monic irreducible polynomials of degree  $m - 1$ . Hence their product, which has degree  $\geq q^{m-1} - q^{m-2}$ , divides  $P$ . Thus  $q^{m-1} - q^{m-2} \leq n$ . We have  $1 - q^{-1} \geq 1/2$ ; thus  $\frac{1}{2}q^m q^{-1} \leq n$ , that is  $q^m \leq 2nq$ .

Some irreducible polynomial of degree  $m$  does not divide  $P$ , hence the quotient provides a field quotient of cardinal  $q^m \leq 2nq$  in which  $P$  survives.  $\square$

**Corollary 2.3.** *Let  $F$  be a field with  $q$  elements and  $P$  a nonzero polynomial in  $F[t_1, \dots, t_k]$ , of degree  $\leq n$  with respect to each indeterminate. Then  $P$  survives in a quotient field of cardinal  $\leq (2n)^k q$ .*

*Proof.* Induction on  $k$ . The result is trivial for  $k = 0$ . Write

$$P = \sum_{i=0}^n P_i(t_1, \dots, t_{k-1}) t_k^i.$$

Some  $P_i$  is nonzero; fix such  $i$ . Then there exists, by induction, some quotient field  $L$  of  $F[t_1, \dots, t_{k-1}]$  of cardinal  $\leq (2n)^{k-1} q$  in which  $P_i$  survives. Then the image of  $P$  in  $L[t_k]$  has degree  $\leq n$  and is nonzero; hence by Lemma 2.2, it survives in a quotient field of cardinal  $2n((2n)^{k-1} q) = (2n)^k q$ .  $\square$

### 3. CONCLUSION OF THE PROOF

**Proposition 3.1.** *Every finitely generated group that is linear over a field of characteristic  $p$  has at most exponential residual girth.*

*Proof.* Such a group embeds into  $\mathrm{GL}_d(K)$  where  $K$  is an extension of degree  $b$  of some field  $K' = F_q(t_1, \dots, t_k)$ , and hence embeds into  $\mathrm{GL}_{bd}(K')$ . Hence it is no restriction to assume that the group is contained in  $\mathrm{GL}_d(F_q(t_1, \dots, t_k))$ . We let  $S$  be a finite symmetric generating subset with 1; it is actually contained in  $\mathrm{GL}_d(F_q[t_1, \dots, t_k][Q^{-1}])$  for some nonzero polynomial  $Q$ .

Write  $S = Q^{-\lambda} T$  with  $\lambda$  a non-negative integer and  $T \subset \mathrm{Mat}_d(F_q[t_1, \dots, t_k])$ ; write  $s = \#(S) = \#(T)$ . If  $x$  is a matrix, let  $b(x)$  be the product of all its nonzero entries (thus  $b(0) = 1$ ). Let  $m$  be such that every entry of every element of  $T$  has degree  $\leq m$  with respect to each variable. Then in  $T^{2n}$ , every entry of every element has degree  $\leq 2nm$  with respect to each variable. Define  $x_n = \prod_{y \in T^{2n}} b(y - 1)$ . Thus  $x_n$  is a product of at most  $d^2 s^{2n}$  polynomials of degree  $\leq 2nm$  with respect to each variable. Define  $x'_n = x_n Q$ ; assume that  $Q$  has degree  $\leq \delta$  with respect to each variable, so that  $x'_n$  has degree  $\leq 2d^2 m n s^{2n} + \delta$  with respect to each variable.

Then, by Corollary 2.3,  $x'_n$  survives in a finite field  $F_n$  of cardinal  $q_1 \leq q(4d^2mns^{2n} + 2\delta)^k$ . Thus  $S^n$  is mapped injectively into  $\mathrm{GL}_d(F_n)$ , which has cardinal

$$\leq q_1^{d^2} \leq q^{d^2} (4d^2mns^{2n} + 2\delta)^{kd^2}.$$

Since  $m, d, k, s, q$  are fixed, this grows at most exponentially with respect to  $n$ .  $\square$

**Proposition 3.2.** *Every finitely generated group that is linear over a field of characteristic 0 has at most exponential residual girth.*

*Proof.* Similarly as in the proof of Proposition 3.1, we can suppose that the group is contained in  $\mathrm{GL}_d(\mathbf{Q}(t_1, \dots, t_k))$ . We let  $S$  be a finite symmetric generating subset with 1; it is actually contained in  $\mathrm{GL}_d(\mathbf{Z}[t_1, \dots, t_k][r^{-1}Q^{-1}])$  for some nonzero integer  $r \geq 1$  and nonzero polynomial  $Q$  with coprime coefficients.

Write  $S = (Qr)^{-\lambda}T$  with  $\lambda$  a non-negative integer and  $T \subset \mathrm{Mat}_d(\mathbf{Z}[t_1, \dots, t_k])$ ; write  $s = \#(S) = \#(T)$ . Let  $R$  be an upper bound on coefficients of entries of elements of  $T$ , and let  $M$  be an upper bound on the number of nonzero coefficients of entries of elements of  $T$ . Then any product of  $2n$  elements of  $T$  is a sum of  $\leq M^{2n}$  monomials, each with a coefficient of absolute value  $\leq R^{2n}$ . Since any entry of an element in  $T^{2n}$  is a sum of at most  $d^{2n-1}$  such products, we deduce that the coefficients of entries of elements of  $T^{2n}$  are  $\leq d^{2n-1}R^{2n}M^{2n}$ . There exists a prime  $p_n \in [2d^{2n-1}(RM)^{2n}, 4d^{2n-1}(RM)^{2n}]$ . There exists  $n_0$  such that for every  $n \geq n_0$ ,  $2d^{2n-1}(RM)^{2n}$  is greater than any prime divisor of  $r$ , and  $2d^{2n-1}(RM)^{2n}$  is greater than the lowest absolute value of a nonzero coefficient of  $Q$ . Now we always assume  $n \geq n_0$ . Then  $S^{2n}$  is mapped injectively into  $\mathrm{GL}_d((\mathbf{Z}/p_n\mathbf{Z})[t_1, \dots, t_k][Q^{-1}])$ .

Let  $m$  be such that every entry of any element of  $T$  has degree  $\leq m$  with respect to each variable. The previous proof provides a quotient  $\mathrm{GL}_d(F_n)$  of  $\mathrm{GL}_d((\mathbf{Z}/p_n\mathbf{Z})[t_1, \dots, t_k][Q^{-1}])$  in which  $S^n$  is mapped injectively, such that  $\mathrm{GL}_d(F_n)$  has cardinal

$$\leq p_n^{d^2} (4d^2mns^{2n} + 2\delta)^{kd^2}$$

Here  $m, d, s, k$  are independent of  $n$ . The latter number is

$$\leq (4d^{-1}(dRM)^{2n})^{d^2} (4d^2mns^{2n} + 2\delta)^{kd^2},$$

which grows at most exponentially with respect to  $n$ .  $\square$

*Proof of Theorem 1.1.* First assume that  $\Gamma$  is linear over some field. By Propositions 3.1 and 3.2, the residual girth, and hence the systolic growth, is at most exponential. If  $\Gamma$  is not virtually nilpotent, then by the Tits-Rosenblatt alternative, it contains a free subsemigroup on 2 generators and hence has exponential growth, and therefore has at least exponential systolic growth and residual girth.

Now assume that  $\Gamma$  is linear over some product of fields. Let  $A$  be the ring generated by entries of  $\Gamma$ . This is a finitely generated reduced commutative ring; hence it has finitely many minimal prime ideals, whose intersection equals the set of nilpotent elements and hence is reduced to zero. Therefore  $\Gamma$  embeds into

a finite product of matrix group over various fields. We conclude that  $\Gamma$  has at most exponential residual girth, using the following two general facts:

- suppose that  $\Gamma_1, \dots, \Gamma_k$  are finitely generated groups and  $\Gamma_i$  has residual girth asymptotically bounded above by some function  $u_i \geq 1$ , then the residual girth of  $\prod_{i=1}^k \Gamma_i$  is asymptotically bounded above by  $\prod u_i$ ;
- if  $\Lambda_1 \subset \Lambda_2$  are finitely generated groups then the residual girth of  $\Lambda_1$  is asymptotically bounded above by that of  $\Lambda_2$ .

□

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